Chaos, Solitons and Fractals 173 (2023) 113724

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

journal homepage: www.elsevier.com/locate/chaos

Wavelets and stochastic theory: Past and future ☆

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ARTICLE INFO

Keywords:
Multiresolution analysis
Time operator
Regular stationary processes
Measurement problem
Optimal representation
Hidden Markov model
Underlying dynamics

ABSTRACT

In the paper, authors report on the interdisciplinary and extremely complex link between wavelets and stochastic processes. An insight into the history of wavelets has been provided presenting the fundamental conception of wavelets, as well as wavelet theory that emerged from stochastic processes. The multiresolution analysis corresponds to the Kolmogorov system which is a regular stationary stochastic process. It presents a significant link to the measurement problem in terms of positional notation which the wavelet domain hidden Markov model should be derived from. The optimal representation arises to be an issue requiring further elaboration extended to the general measurement and wavelet frames.

1. Introduction

Wavelets lie on the boundary of diverse scientific areas, such as mathematics, physics, signal processing, and image processing. The aim of wavelet studying is to give a coherent theory that should resolve common difficulties which are met in various disciplines mentioned above [1]. Beside providing continuous advances in mathematical methods [2] and representation of nonstationary signals [3–5], wavelets are successfully used in a wide range of applications [6–8]. Spanning in a variety of scientific and engineering fields, wavelets have been recently applied to cutting-edge topics such as the outbreak of the coronavirus [9,10], economic growth and stock analysis [11–13], modeling and analysis of biomedical signals and systems [14,15], characterization of nonlinear geophysical signals [16,17], information processing in control stochastic systems [18], etc. The fact of concurrent emergence in widely differing domains indicates that wavelet theory is a common ground for all of them and mathematics has largely benefitted on the rediscovery of wavelets by experts from other disciplines [1]. The above motivates the presented elucidation of an extremely complex and interdisciplinary link to stochastic theory.

The usage of wavelets (in French *ondelette* [19] – little wave) spread widely due to more than half a century long synchronous and independent activities in mathematics, physics and signal processing. Their employment has been inaugurated in 1909 in the dissertation of the Hungarian mathematician Alfred Haar (1885–1933), entitled *Zur Theorie der orthogonalen Funktionensysteme*, supervised by David Hilbert, and published a year later in the *Mathematische Annalen* [20]. He proposed a complete orthonormal system that is not regular in terms of continuous differentiability, which was produced via translation and dilation of the Haar wavelet [21].

Diverse efforts that were present up to the 1980s include triangle functions defined by Schauder and Faber in 1910–1920, the Franklin orthonormal system from 1927, the Littlewood-Paley theory from about 1930, multifractal structure of Brownian motion analyzed by Lévy in 1930s, time–frequency Gabor atoms in signal processing from 1946, the Calderón identity from 1960, subband coding by Croisier, Esteban and Galand in 1975, continuous wavelet transform by Zweig in 1975, geometry of Banach spaces given by Strömberg in 1981, pyramidal algorithms for image processing by Burt and Adelson in 1982, Marr's zero-crossings in human vision from 1982, spline approximations, the Rokhlin multipole algorithms from 1985, refinement schemes in the computer graphics, coherent states in quantum mechanics, and renormalization in quantum field theory [22,23]. A multitude of them has been consolidated during the 1980s.

Cooperation of mathematicians and physicists with geologists was the key factor for the synthesis of previously independent achievements. Alexander Grossmann (1930–2019) and Jean Morlet (1931–2007) defined wavelets to be some translations and dilations which are usable for a series expansion of the signal, and Yves Meyer (1939–) presented a wavelet which is a generalization of the Haar one [24]. French geophysicist Morlet was working for an oil company within

☆ This paper is an extended version of our abstract published at the Book of Abstracts of The Seventh Conference on Information Theory and Complex Systems, TINKOS 2019, Belgrade, Serbia, October 15–16, 2019.

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https://doi.org/10.1016/j.chaos.2023.113724
Received 25 April 2023; Received in revised form 13 June 2023; Accepted 14 June 2023
Available online 5 July 2023

which he searched for the mathematical tool that should process seismic signals and get information about geological layers [25]. Since the Fourier transform has been considered inappropriate, he proposed to use dilations and translations of a wavelet [22,26] and developed a cycle-octave representation having turned the time–frequency into the time-scale analysis [27]. Grossmann was a theoretical physicist and the mentor of investigating coherent states by Belgium mathematician Ingrid Daubechies (1954–) who constructed wavelets that had compact support and arbitrary high regularity [28]. Daubechies, Grossmann and Morlet were part of a distinguished Marseille "Club of waveletters" which published the most influential articles on wavelet theory in the 1980s. The number of papers had been permanently increasing, so that there were identified over a thousand by the end of 1998 [29].

During the 1980s, syntheses of various disciplines brought impressive achievements. The first one resulted in wavelet analysis and the second in multiresolution analysis [30]. A French student in computer vision Stéphane Mallat (1962–) recognized similarities between wavelets and algorithms that had been used in image processing (pyramid algorithms), signal processing (subband coding), and digital speech processing (quadrature mirror filters) [23,31]. Due to his work with French mathematician Meyer, multiresolution analysis became an essential tool in the study of wavelets [32,33]. It can be intended as the bridge connecting the digital signal on one side and the continuous wavelet transform on the other [21].

An American mathematician Karl Gustafson (1935–) has presented a view in which wavelets are regarded to be stochastic processes [34]. The theory arose from the time operator formalism of complex systems which had originated by the Brussels school of thermodynamics [35]. Together with the Indian physicist Baidyanath Misra (1937–), Gustafson looked at models for the decay of quantum particles, having realized that the regular stationary stochastic process implies multiresolution property which was an indication of the time operator [36]. Undoubtedly, quantum physics played a significant role for the formulation of wavelet theory [34]. Though Daubechies had studied coherent states, there was no relation established to multiresolution analysis and stochastic processes [37].

Multiresolution analysis, introduced by Meyer and Mallat, corresponds to the theory of Kolmogorov systems which belong to the framework of regular stationary stochastic processes [38]. A paradigmatic instance of the Kolmogorov automorphism has been induced by baker‘s map, being a measure preserving transformation of the unit square [39]. A significant link to the measurement problem in terms of positional notation elucidates a relationship to the optimal representation of a stochastic process [40]. In that regard, wavelet theory concerns a perceptual psychophysics which is aimed at acquiring structural information that the measurement should refer to [41]. The wavelet domain hidden Markov model recapitulating statistical properties of the process has been proven tremendously useful in a variety of applications [42]. Signal processing procedures such as compression and denoising perform at best in respect to the optimal base [43]. A further elaboration extending it to the general measurement and wavelet frames has been required [44].

The current paper considers that stochastic theory is an adequate framework presenting wavelets within, which has already been corroborated by notable attempts [45]. After the Introduction, Section 2 has exposed the fundamental conception of wavelets. Section 3 should present wavelet theory emerging from stochastic processes. Results are discussed in Section 4 and the last one contains concluding remarks.

2. Fundamental conception

2.1. Wavelet transform

From the early attempts of signal processing, it appeared clear that the Fourier analysis failed to achieve the optimal representation. First of all, the Fourier transform is not compatible with real-time transmission since integrating a function over all space. It often happens however that the function is only defined on an open domain or transmitted through the timeline expansion. If one extends the function by a null outside the domain, it should cause jump discontinuities across the boundary and the Fourier transform of the artificial function might be more sensitive to them than to the intrinsic properties of a signal inside the domain [46].

The problem was less evident before computers became common and began to play significant role in applied mathematics. Danish mathematician Palle Jorgensen (1947–) has summed up computational drawbacks of the Fourier transform in a sonant buzzword: "Computers can't integrate" [45]. But the point is even more problematic if one realizes that computers do not handle anything that should be integrated. An extensional concept of the function taking a nonenumerable set of real (or complex) values over the nonenumerable set of real arguments is merely useless in terms of any computation, which has impelled one to proceed the data in a more pragmatic fashion [47]. Considered the above, both the Fourier transform and the function it applies to, are useless on the matter of computation, which brings the representational issue to the fore.

John von Neumann (1903–1957), Dennis Gabor (1900–1979), Leon Brillouin (1889–1969), Eugene Wigner (1902–1995), Claude Shannon (1916–2001) et al. who had pioneered signal processing in the 1940s, addressed the issue and advocated for some solutions [46]. But it was not so easily resolved, demanding a synchronous effort in various areas. The main advantage that was presented by wavelets concerns the octave conception, which is a dyadic scale replacing frequency in the Fourier analysis. Normalized scaling of a *mother wavelet* ψ has supplemented by the spatial translation, in order to produce a wavelet $\psi_{j,k} = 2^{nj/2}\psi(2^j \cdot -k)$ which is the structural element of such a representation. Detail coefficients $D_{j,k} = \langle \psi_{j,k} | f \rangle$ should reproduce the signal due to a decomposition $f = \iint dj\ dk\ D_{j,k}\psi_{j,k}$ which is termed the Calderón identity. It holds if the mother wavelet ψ has satisfied the admissibility condition $\int dj\ |\hat{\psi}(2^j\xi)|^2 = 1$ almost everywhere. If the mother belongs to $L^2_{m^n}$, wherein m^n is the Lebesgue measure over $\mathbb{R}^n$, the reproducing identity is valid for all signals from the Hilbert space [48].

If ψ belongs to $L^1_{m^n}$, $\hat{\psi}$ is continuous and the admissibility implies $\hat{\psi}(0) = 0$ meaning that $\int dm^n\ \psi = 0$, i.e., the wavelet is a null-integrable oscillatory function. Such a condition is nearly sufficient as well. If the null-integrability holds and $\hat{\psi}$ is continuously differentiable, ψ is admissible up to a constant factor and it has a sufficient decay [49].

The representation of a signal f via detail coefficients $\int dm^n\ \psi^*_{j,k} f$, which is termed the continuous wavelet transform that involves every real-valued scaling and every $\mathbb{R}^n$-translation of the mother wavelet, might be highly redundant. A discrete refinement should imply integer indices which concern the reproducing identity $f = \sum_{j,k} D_{j,k}\psi_{j,k}$. If the redundancy is eliminated in that manner, it is about orthonormal or biorthonormal wavelets which are some bases of the Hilbert space. If otherwise, the mother wavelet has produced a redundant frame.

2.2. Orthonormal wavelets

Considering orthonormal bases of $L^2_{m^n}$, one should suppose the norm $\|\cdot\|$ which is induced by the inner product $\langle\cdot|\cdot\rangle$ of the Hilbert space. In that respect, orthonormal wavelets satisfy $\langle \psi_{j,k} | \psi_{l,m} \rangle = \delta^{l,m}_{j,k}$. The right-hand side exhibits the Kroneker delta symbol that has a null value if upper and lower indices differ and a unit otherwise. The requirement is in accordance to the property $\psi_{j,k} = 2^{nj/2}\psi(2^j \cdot -k)$ that has produced wavelets from the mother ψ.

Orthonormal wavelets refer to a specific construction in the Hilbert space which is based upon a scale self-similarity giving rise to the cascade family of approximation subspaces. The multiresolution analysis is a term derived from optics and computer vision [30,33], which concerns the sequence V_j satisfying axioms:

(i) it is increasing, i.e., $V_j \subseteq V_{j+1}$;
(ii) the supremum is dense, i.e., $\overline{\cup_j V_j} = L^2_{m^n}$;
(iii) the infimum consists of constants only, i.e., $\cap_j V_j = \mathbb{0}$;
(iv) it evolves due to the shift operator $U : f(\cdot) \mapsto f(2\cdot)$, i.e., $f \in V_j \Leftrightarrow Uf \in V_{j+1}$;
(v) the translation of a scaling function ϕ by integers should produce $\phi_k = \phi(\cdot - k)$ which is the orthonormal base of the approximation subspace V_0.

The relation between wavelets and a multiresolution analysis is given by the sequence W_j which are orthogonal complements of V_j in V_{j+1}. If translating the function ψ by integers gives $\psi(\cdot - k)$ which is an orthonormal base of W_0, it is regarded to be the mother of orthonormal wavelets. A higher dimension n comes down to one-dimensional multiresolution analysis due to the tensor factorization $L^2_{m^n} = \underbrace{L^2_m \otimes \cdots \otimes L^2_m}_{n}$, that has allowed 2^n variations $\underbrace{\begin{matrix}\phi \\ \psi\end{matrix} \otimes \cdots \otimes \begin{matrix}\phi \\ \psi\end{matrix}}_{n}$, of which the first one corresponds to the scaling function $\underbrace{\phi \otimes \cdots \otimes \phi}_{n}$, and other ones to $2^n - 1$ mother wavelets ψ^i. The reproducing identity is a decomposition $f = \sum_{i,j,k} D^i_{j,k} \psi^i_{j,k}$, wherein $D^i_{j,k} = \left\langle \psi^i_{j,k} \middle| f \right\rangle$ are detail coefficients and $\psi^i_{j,k} = 2^{nj/2} \psi^i(2^j \cdot - k)$ are wavelets which are produced by their mothers [49].

In regard to the one-dimensional construction, the mother wavelet ψ is immediately built from the scaling function ϕ which is termed *father wavelet*. A multiresolution analysis of the Hilbert space $L^2_m = \bigoplus_j W_j$ is representable in terms of detail subspaces W_j, which are generated by wavelets $\psi_{j,k} = 2^{j/2} \psi(2^j \cdot - k)$. The converse issue concerns the matter whether any orthonormal wavelet might be constructed from a multiresolution analysis. It is true only if the mother satisfies certain properties on regularity and localization [50].

A prototype of mother wavelet is the Haar one $\chi(\omega) = \begin{cases} +1, & 0 \le \omega < \frac{1}{2} \\ -1, & \frac{1}{2} \le \omega < 1 \end{cases}$ implying a null value otherwise, which stems from the father $\phi = \chi^2$ that is an indicator of the unit interval $\mathbb{I} = [0, 1)$. The prototypicality means that some statements concerning the Haar wavelet are almost satisfied by anyone else, such as $|\psi|^2 \approx 1$ inside $\mathbb{I}$ or $\psi \approx 0$ outside of it. The symbol $\approx$ in that respect should designate the equality that is approximative for any wavelet, although it strictly holds for an instance of χ only.

A scaling function $\phi \approx \chi^2$ of the multiresolution analysis has produced a base $\phi_k \approx \chi^2(\cdot - k)$ of the approximation subspace V_0. According to that, the decomposition $f_0 = \sum_k A_k \phi(\cdot - k)$ means that the approximation coefficients should satisfy $f_0 \approx A_{\lfloor \cdot \rfloor}$ wherein $\lfloor \cdot \rfloor$ denotes the floor function. A signal f_0 from V_0 approximately depends on the integer part of an argument only, which transmits to all subspaces V_j that are approximately independent of digits after the j-th one counted from the binary point on. Since an integer k corresponds to $k + \mathbb{I} = [k, k+1)$, $\mathbb{Z}$ is coincident to the σ-algebra generated by such intervals. In that manner, $\mathbb{Z}_j = \frac{\mathbb{Z}}{2^j}$ has defined an increasing sequence that approximately follows the multiresolution analysis, involving dependence on one more binary digit of the domain in each step. If f_j is the orthogonal projection of the signal f onto the approximation subspace V_j, it holds the estimate $f_j \approx f|_{\mathbb{Z}_j}$ which is the conditional expectation over the interval σ-algebra $\mathbb{Z}_j$. Substracting the approximation, one gets $f_{j+1} - f_j = \sum_k D_{j,k} \psi_{j,k}$ which is the orthogonal projection of the signal onto the detail subspace W_j. In respect to a multiresolution analysis, the reproducing identity $f = \sum_{j,k} D_{j,k} \psi_{j,k}$, has decomposed a signal via successive scales each approximating dependence on one bit of the argument.

2.3. Wavelet frames

Orthonormal wavelets $\psi_{j,k}$ are bases of the Hilbert space, whose reproducing identity $f = \sum_{j,k} D_{j,k} \psi_{j,k}$ implies detail coefficients of a signal f to be $D_{j,k} = \left\langle \psi_{j,k} \middle| f \right\rangle$. Such a decomposition is equivalent to the Parseval identity $\|f\|^2 = \sum_{j,k} |\langle \psi_{j,k} | f \rangle|^2$, which should have been valid for each signal f. If the inequality $A\|f\|^2 \le \sum_{j,k} |\langle \psi_{j,k} | f \rangle^2| \le B\|f\|^2$ holds, $\psi_{j,k}$ is the *wavelet frame* for some positive constants A and B which are termed *frame bounds*.

The frame might not be a base of the Hilbert space, or linearly independent at all. But it must be the generator, since if not there is a signal f orthogonal to each of elements. It follows that $A\|f\|^2 = \sum_{j,k} |0|^2$, which means that such a signal is the null only. The frame is termed to be *tight* if its bounds are equal. It is the *Parseval* one if both bounds are equal to 1, which implies the reproducing identity $f = \sum_{j,k} \left\langle \psi_{j,k} \middle| f \right\rangle \psi_{j,k}$. The Parseval frame $\psi_{j,k}$, which is a base of the Hilbert space coincides to orthonormal wavelets [51].

$\widetilde{\psi_{j,k}}$ is a dual frame of $\psi_{j,k}$ if the reproducing identity $f = \sum_{j,k} \left\langle \psi_{j,k} \middle| f \right\rangle \widetilde{\psi_{j,k}}$ is valid, which has reconstructed a signal f from detail coefficients $D_{j,k} = \left\langle \psi_{j,k} \middle| f \right\rangle$. It is termed to be the *canonical dual* if $\widetilde{\psi_{j,k}} = \mathfrak{S} \psi_{j,k}$ which means that the operator mapping one frame to the other implies $\mathfrak{S}^{-1} : f \mapsto \sum_{j,k} \left\langle \psi_{j,k} \middle| f \right\rangle \psi_{j,k}$. For any invertible operator $\mathfrak{F}$ such that $\mathfrak{F}\psi_{j,k}$ corresponds to the Parseval frame, the factorization $\mathfrak{S} = \mathfrak{F}^\dagger \mathfrak{F}$ holds. If the frame $\psi_{j,k}$ is a base of the Hilbert space, there is only a canonical dual $\widetilde{\psi_{j,k}}$ that has satisfied $\left\langle \psi_{j,k} \middle| \widetilde{\psi_{l,m}} \right\rangle = \delta^{l,m}_{j,k}$. In that instance, $\psi_{j,k}$ and $\widetilde{\psi_{j,k}}$ are biorthonormal wavelets which means that the dual frame is also produced by a mother $\widetilde{\psi}$. The frame $\psi_{j,k}$, having bounds A and B, should satisfy $A \le \sum_j \left|\hat{\psi}(2^j \xi)\right|^2 \le B$, which is a relaxed admissibility that applies to continuous wavelet transform as well [52]. The reproducing identity in terms of a dual frame $f = \iint dj\, dk\, D_{j,k} \widetilde{\psi_{j,k}}$, which has implied detail coefficients $D_{j,k} = \left\langle \psi_{j,k} \middle| f \right\rangle$, requires the integral $\int dj \left| \hat{\psi}\left(2^j \xi\right) \right|^2$ to be bounded by positive constants almost everywhere [53].

A converse statement in general involves some more detail. One defines $S(\xi) = \sum_j \left|\hat{\psi}(2^j \xi)\right|^2$, $t_q(\xi) = \sum_{j \ge 0} \hat{\psi}\left(2^j \xi\right) \hat{\psi}^*(2^j(\xi + 2(2q-1)\pi))$ and $\beta(q) = \text{supess} \sum_k t_q(2^k \xi)$. A wavelet ψ is the mother of a frame having bounds A and B if

$$A = \text{infess}\, S - \sum_q \sqrt{\beta(q)\ \beta(-q)}$$

and

$$B = \text{supess}\, S + \sum_q \sqrt{\beta(q)\ \beta(-q)}$$

are positive constants. The mother has produced a tight frame $\psi_{j,k}$, if $\sum_j \left|\hat{\psi}(2^j \xi)\right|^2 = 1$ and $\sum_{j \ge 0} \hat{\psi}\left(2^j \xi\right) \hat{\psi}^*(2^j(\xi + 2(2q-1)\pi)) = 0$ almost everywhere for each integer q. $\psi_{j,k}$ are orthonormal wavelets if the mother is moreover normalized [54].

3. Results

3.1. Wavelets on the interval

In order to present a stochastic theory of wavelets, one requires the probability μ, which is a restriction of the Lebesgue measure m onto $\mathbb{I} = [0, 1)$. Orthonormal wavelets on the unit interval arise from those on the real line $\mathbb{R} = \bigcup_i i + \mathbb{I}$ via a periodization $\overline{\psi_{j,k}} = \sum_i \psi_{j,k}(i + \cdot)$. In the same manner, a multiresolution analysis is periodized into the increasing sequence of approximation subspaces $\overline{V_j}$ and detail subspaces $\overline{W_j}$ which are wandering by a unilateral shift $U : f(\cdot) \mapsto f(R\cdot)$ that has been induced by the Rényi map $R\omega = \begin{cases} 2\omega, & 0 \le \omega < \frac{1}{2} \\ 2\omega - 1, & \frac{1}{2} \le \omega < 1 \end{cases}$.

Since $\psi \approx 0$ out of $\mathbb{I}$, wavelets on the interval approximate those on the real line $\overline{\psi_{j,k}} \approx \psi_{j,k}$ for any $j \ge 0$, which applies as well to approximation and detail subspaces. That is the reason for using the symbol $\psi_{j,k}$ henceforward implying wavelets on the interval, and the same for multiresolution analysis. Considering the adopted notation, the annihilation property $\psi_{j,k} = 0$ holds whenever $j < 0$, which also

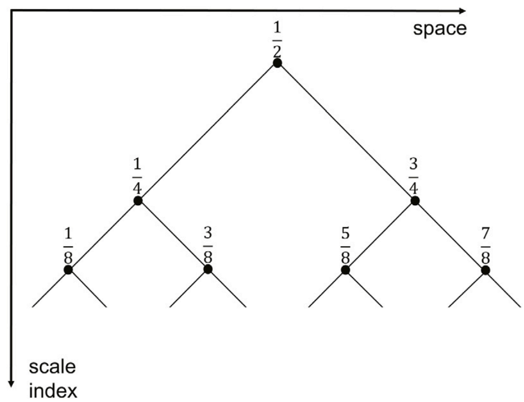

Fig. 1. Binary tree of detail coefficients. Each scale corresponds to one bit of the unit interval.

means $V_0 = \mathbb{0}$, i.e., approximation subspaces having the null or the negative index consist of constants only [54].

In that respect, orthonormal wavelets on the unit interval should satisfy the axioms:

(i) annihilation, i.e., $j < 0 \Rightarrow \psi_{j,k} = 0$;
(ii) periodization, i.e., $\psi_{j,k} = \psi_{j,k+2^j}$;
(iii) translation, i.e., $\psi_{j,k+m} = \psi_{j,k}(\cdot - \frac{m}{2^j})$;
(iv) evolution, i.e., $U^\dagger \psi_{j,k} = \frac{1}{\sqrt{2}}\psi_{j-1,k}$;
(v) orthonormality, i.e., for $j \geq 0$ and $1 \leq k \leq 2^j, \psi_{j,k}$ is an orthonormal base of $L^2_\mu \ominus \mathbb{0}$, which is an orthocomplement of constants in the Hilbert space.

According to that, the reproducing identity $f = A + \sum_{j,k} D_{j,k}\psi_{j,k}$ holds on L^2_μ implying summation over basic elements. From the axiom (v) it follows that $\psi_{j,k}$ are decorrelated random variables having null expectation and unit variance. An immediate consequence of the axiom (iii) is that such variables are equally distributed within each scale. The axiom (iv) is a concise note on the evolution $U\psi_{j,k} = \frac{1}{\sqrt{2}}\psi_{j+1,k}\frac{1}{\sqrt{2}}\psi_{j+1,k+2^j}$ in terms of an adjoint operator $U^\dagger : f(\cdot) \mapsto \frac{f\left(\frac{\cdot}{2}\right)+f\left(\frac{\cdot+1}{2}\right)}{2}$ which is the left inverse of U, i.e., $U^\dagger U = I$. Therefore U is an L^2_μ-isometry, which is implied by the measure preserving map R that has induced it. Although the adjoint operator $U^\dagger$ is not an isometry, it preserves the L^1_μ-norm of the wavelet $\psi_{j,k}$ which is evident in (iv). In that regard, wavelets on the unit interval are accommodated in L^1_μ, which involves the Hilbert space L^2_μ.

The adjoint $U^\dagger$ is the Frobenius-Peron operator governing the evolution of distribution densities. The observation should imply that wavelets correspond to some densities in a general manner, which is manifested by the absolute square $\left|\psi_{j,k}\right|^2$ being L^1_μ-normalized. The expectation of a variable $\Omega_{j,k}$, that is distributed according to $\left|\psi_{j,k}\right|^2$, approximates to $E\Omega_{j,k} \approx \frac{2k-1}{2^{j+1}}$, which represents a link between wavelets and nodes of the binary tree. Each scale of details in a multiresolution analysis involves basic elements corresponding to one binary digit. In that respect, the inheritance along branches of the binary tree is related to the refinement of the resolution that approximately depends on one more bit of the unit interval (Fig. 1).

3.2. Exact and the Kolmogorov system

Yves Mayer has pointed out that "one cannot expect any serious understanding of what wavelet analysis means without a deep knowledge of corresponding operator theory" [46]. An advantage of using the Koopman operator to present the evolution of a system is the avoidance of point trajectories in order to distinguish a common behavior [55]. The behavior is not affected by particular points which are null-measured, but composite domains only. The observation is coincident to a statement on the epistemological value of probability theory, which concerns "large-scale random phenomena creating strict non-random regularity" [56]. Ergodic properties of a system indicate the manner such a task should be realized.

The Koopman operator U governs the evolution of the random variable preserving its distribution, since it has been induced by the Rényi map which preserves the probability measure $\mu\ R^{-1} = \mu$. It has generated the exact system, meaning $\mu(A) \neq 0 \;\Rightarrow\; \lim_{j\to\infty}\mu(R^j A) = 1$, i.e., any non-null domain is rising to the full measure. Exactness implies mixing, that is an ergodic property of systems which provide the time operator formalism [57]. Supposing wavelets $\psi_{j,k}$ on the unit interval, there is an operator that satisfies $T\psi_{j,k} = j\psi_{j,k}$, which is defined on the dense subspace of $L^2_\mu \ominus \mathbb{0}$ [58]. From the axiom (iv), one derives $\left[U^\dagger,\ T\right]\psi_{j,k} = U^\dagger T\psi_{j,k} - TU^\dagger\psi_{j,k} = jU^\dagger\psi_{j,k} - (j-1)\,U^\dagger\psi_{j,k} = U^\dagger\psi_{j,k} \Rightarrow [T,U] = U$, which is a defining feature of the time operator [59].

It is obvious that the exact system which preserves measure is not generated by an invertible map. According to that, R is non-invertible and $R^{-1}A = \frac{A}{2} \cup \frac{A+1}{2}$ denotes the inverse image of the domain. The same holds for U, which is not a unitary operator just because there is no inverse. However, an exact system extends to the Kolmogorov one which is invertible [57]. The natural extension of R concerns baker‘s map $B(\omega_1,\varpi) = \begin{cases} \left(2\omega_1, \frac{\varpi}{2}\right), & 0 \leq \omega_1 < \frac{1}{2} \\ \left(2\omega_1 - 1, \frac{\varpi+1}{2}\right), & \frac{1}{2} \leq \omega_1 < 1 \end{cases}$, which is a bilateral shift preserving the measure $\mu^2 = \mu \otimes \mu$. The shift occurs via binary digits of an argument which might be written in the form $\varpi.\omega = \cdots\omega_{-1}.\omega_0\omega_1\ldots$ wherein ω_j are bits of ω if $j \geq 0$ or of ϖ if not. Baker‘s map shifts binary point rightwards according to $B\varpi.\omega = \cdots\omega_{-1}\omega_0.\omega_1\ldots$

Multiresolution analysis of the Haar wavelet presenting dependence on binary digits is naturally extended, which involves a shift of approximation subspaces $\overline{\overline{V^\chi_j}}$ by the operator $U^\chi : f(\cdot) \mapsto f(B\cdot)$. Each of them includes an approximation subspace V^χ_j over the real line, but it goes beyond, since $\overline{\overline{V^\chi_0}}$ consists of signals over dyadic integers $\overline{\overline{Z}}$ and similar for other ones which are over $\overline{\overline{\mathbb{Z}_j}} = \frac{\overline{\overline{\mathbb{Z}}}}{2^j}$. In accordance, the union $\overline{\overline{\bigcup_j\ V^\chi_j}} = L^2_h$ corresponds to the Hilbert space that implies the Haar measure h which is defined over the dyadic line $\mathbb{D} = \overline{\overline{\bigcup_j \mathbb{Z}_j}}$. Such a multiresolution analysis over dyadics concerns approximation subspaces $\overline{\overline{V^\chi_j}}$ only, in respect to the shift operator $f(\cdot) \mapsto f(2\cdot)$ which is induced by a natural scaling of the dyadic line [60]. The denotation $\overline{\overline{\cdot}}$ should mean the closure in regard to the dyadic norm, which is related to the floating point representation [61].

The operator U^χ, that is induced by baker‘s map, is not a single extension of the unilateral shift U onto the Hilbert space $L^2_{\mu^2} = L^2_\mu \otimes L^2_\mu$, which is invertible. The base exchange $C : \chi_{j,k} \otimes \chi_{l,m} \mapsto \psi_{j,k} \otimes \psi_{l,m}$ defines the operator $U = CU^\chi C^{-1}$ extending the previous one, which is a bilateral shift of the multiresolution analysis for wavelets $\psi_{j,k}$ [40]. As well as in the real line instance, it approximates the dependence on bits of the domain which is presented by the Haar one. In that respect, statements $\overline{\overline{V_j}} \approx \overline{\overline{V^\chi_j}}$ and $U \approx U^\chi$ concern an attempt to present approximation subspaces and the evolutionary operator in terms of positional notation. Regardless of the fact that it does not strictly hold, the approximation gives an insight into the concept of wavelet theory that relates to the measurement problem.

3.3. Measurement problem

The von Neumann measurement corresponds to a complete system of mutually orthogonal projectors on the Hilbert space [62]. Supposing orthonormal wavelets $\psi_{j,k}$, such projections are regarded to be $P_{j,k} =$

$\left\langle \psi_{j,k} \middle| \cdot \right\rangle \psi_{j,k}$ which satisfy the expression $\sum_{j,k} P_{j,k} = I$ on $L^2_\mu \ominus \mathbb{O}$. The inner product of operators is defined $\left\langle P \middle| Q \right\rangle = \mathrm{tr}\ P^\dagger Q$.

A quantum ensemble, which the measurement acts onto, is a map $M : P \mapsto \left\langle \rho \middle| P \right\rangle$ assigning a probability to each projector. According to the Gleason theorem, for any ensemble there is a density operator $\rho \geq 0$, which means that $\rho = FF^\dagger$ whereat F is a root. Due to probability normalization, it holds $\mathrm{tr}\ \rho = 1$, which has implied $\|F\| = 1$ in terms of $L^2_{\mu^2}$-norm.

Because of commutativity, projectors $P_{j,k}$ generate a σ-algebra $\mathfrak{M}$ following the Stone representation theorem. It is a measurable space over which the observable has been defined. The probability measure, corresponding to the density operator ρ, should be defined over the measurable space as well, which implies the commutativity $P_{j,k}\rho = \rho\ P_{j,k}$.

The measurement is therefore related to a superprojection $\mathcal{M}\rho = \sum_{j,k} \underbrace{\left\langle P_{j,k} \middle| \rho \right\rangle}_{M(P_{j,k})} P_{j,k}$ onto the subspace of operators satisfying the commutativity rule. The optimal measurement $\rho = \mathcal{M}^o\rho$ has been performed in respect to eigenprojectors of the density operator, which means it measures the probability itself. In that instance, the measurement procedure has actually been intended to stipulate probabilities $M(P_{j,k})$ which are eigenvalues of the density operator [47].

The measurement is an irreversible process, due to the superprojection onto variables over a measurable space. The situation is elegantly presented by von Neumann, who has figured out an entropy increase of the process. He was wondered by the fact that the increase of entropy did not imply any temporal evolution, as it is evident in thermodynamics. The reason is a misleading concept of time which is reduced to the linear parametrization, unlike observables that are represented by Hermitian operators whose eigenprojectors generate a measurable space. The time operator should therefore be a resolution of the measurement problem [62].

The measurable space $\mathfrak{M}_j$, which is generated by projectors $P_{<j,k}$ whose first index is smaller than j, is a subalgebra of $\mathfrak{M}$. If F_j is the orthogonal projection of F to the approximation subspace $\overline{\overline{V_j}}$, it holds $F_j = F|_{\mathfrak{M}_j}$ which represents the conditional expectation over the σ-algebra. On each subalgebra, one assigns measures $M_j = \left\langle \rho_j \middle| \cdot \right\rangle$ to projected densities $\rho_j = F_j F_j{}^\dagger$, which are dependent on $P_{<j,k}$ only. Having corresponded a projector $P_{j,k}$ to the dyadic interval $\frac{2k-1}{2^{j+1}} + \mathbb{Z}$, that concerns its position in the binary tree, there appears a domain of the σ-algebra $\mathfrak{M}$, which has involved a dyadic line $\mathbb{D}$. In that regard, the measurable space $\mathfrak{M}_j$ is constructed over the domain $\overline{\overline{\mathbb{Z}_j}} = \frac{\overline{\overline{\mathbb{Z}}}}{2^j}$, representing independence of digits after the jth one counted from the binary point on.

The decomposition $\sum_{j,k} P_{j,k} = I$ gives rise to a reproducing identity $F = \sum_{j,k} \psi_{j,k} \otimes D_{j,k}$, which is valid on $L^2_\mu \ominus \mathbb{O} \otimes L^2_\mu \ominus \mathbb{O}$, since it holds $P_{j,k}F = \psi_{j,k} \otimes D_{j,k}$. The optimal measurement $\mathcal{M}^o\rho = \sum_{j,k} \left\langle \rho \middle| P^o_{j,k} \right\rangle P^o_{j,k}$ should satisfy $\rho = \mathcal{M}^o\rho$ in terms of a density operator $FF^\dagger = \sum_{j,k,l,m} ED^o_{j,k} D^{o*}_{l,m} \psi^o_{j,k} \otimes \psi^{o*}_{l,m}$. From $P^o_{j,k} = \psi^o_{j,k} \otimes \psi^{o*}_{j,k}$ it follows $ED^o_{j,k} D^{o*}_{l,m} = 0 = ED^o_{j,k} ED^{o*}_{l,m}$ whenever $(j,k) \neq (l,m)$, which means that detail coefficients of an ensemble are decorrelated considering the base of optimal wavelets $\psi^o_{j,k}$. The statistical model of a measurement process regards the detail coefficient to be a random variable, which is achieved extending the Hilbert space L^2_μ to $L^2_\mu \otimes L^2_\mu$. Eigenvalues of the density operator $\rho = FF^\dagger$ correspond to probabilities $M\left(P^o_{j,k}\right) = E\left|D^o_{j,k}\right|^2$.

In another base that is suboptimal, detail coefficients should take the form $D_{j,k} = \sum_{l,m} \left\langle \psi_{j,k} \middle| \psi^o_{l,m} \right\rangle D^o_{l,m}$ which implies an approximate decorrelation of the ensemble. Considering $\psi_{j,k} \approx 0$ out of the interval $[\frac{k-1}{2^j}, \frac{k}{2^j})$, as well as $\psi^o_{l,m}$ out of $[\frac{m-1}{2^l}, \frac{m}{2^l})$, inner products $\left\langle \psi_{j,k} \middle| \psi^o_{l,m} \right\rangle$ are negligible if approximate supports do not intersect. It means that correlation between detail coefficients $\mathbf{D} = \left(D_{j,k}\right)$ predominantly concerns an inheritance along branches of the binary tree. In that manner, one obtains the wavelet domain hidden Markov model which has been proven tremendously useful in a variety of applications including speech recognition and artificial intelligence. The model claims that the correlation is realized through hidden variables $\mathbf{S} = \left(S_{j,k}\right)$, which are attributed to each node forming the Markovian tree, and out of such an interdependence the ensemble is decorrelated (Fig. 2). The conditional distribution $\mathbf{D}|\mathbf{S}$ is supposed to be normal, whence it follows that $D_{j,k}|S_{j,k}$ are independent variables [42].

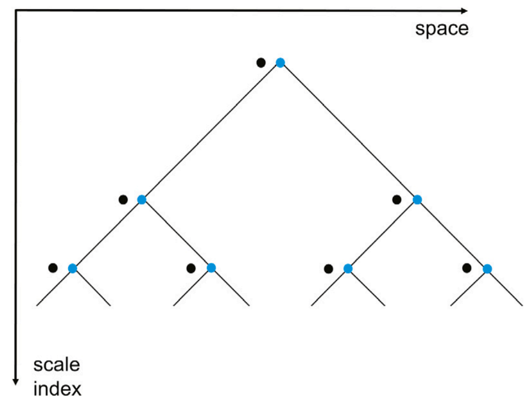

Fig. 2. The wavelet domain hidden Markov model. Black nodes correspond to detail coefficients and blue ones to hidden variables. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Projectors $P_{j,k}$ evolve by the superopeator $\mathcal{U}P = UPU^\dagger$ in the manner of $\mathcal{U}P_{j,k} = P_{j+1.k} + P_{j+1,\ k+2^j}$, which corresponds to the wandering of detail subspaces in the multiresolution analysis by the evolutionary operator U. Since $M\left(\mathcal{U}P\right) = \left\langle \mathcal{U}^\dagger\rho \middle| P \right\rangle$, such an evolution is coincident to the adjoint $\mathcal{U}^\dagger\rho = U^\dagger\rho\, U$ which is the Frobenius–Perron superoperator acting on a density. The time operator T satisfying $[T, U] = U$ is applied to $\mathcal{U}$ as well, since $[T, \mathcal{U}]\, P = TUPU^\dagger - UTPU^\dagger = [T, U]\, PU^\dagger = \mathcal{U}P \Rightarrow [T, \mathcal{U}] = \mathcal{U}$. Its existence makes possible a change in representation $\Lambda = \lambda(T)$ that is an operator function of time, which has transfigured the reversible evolution by $\mathcal{U}^\dagger$ into an irreversible one by $\mathcal{W}^\dagger = \Lambda\mathcal{U}^\dagger\Lambda^{-1}$. Although there is an inverse superoperator $\mathcal{W}^{\dagger-1} = \Lambda\mathcal{U}\Lambda^{-1}$, it does not preserve positivity, which is required to map one density to another [39]. The irreversibility corresponds to a Markov process which is presented by hidden variables of the statistical model [44].

4. Discussion

4.1. Optimal representation

The concept of measurement was presented in Euclid's *Elements*, whose the V book elaborated the doctrine on proportion concerning commensuration of magnitudes. The commensuration procedure is termed the *Euclidean algorithm*, which is a process taking place step by step over time. It should result in a continued fraction expansion, which is the primordial code of a measurement.

Donald Knut called Euclide's method "the granddaddy of all algorithms, because it is the oldest nontrivial algorithm that has survived to the present day" [63]. But it is even more — an intensional procedure which has been producing real numbers of the unit interval. The automorphism

$$? : \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{\ddots}}} \mapsto \frac{1}{2^{n_1 - 1}} - \frac{1}{2^{n_1 + n_2 - 1}} + \cdots$$

which is termed the question mark function by Minkowski, has transfigured the continued fraction to the binary code $\omega = \underbrace{0.0\ldots0}_{n_1}\underbrace{1\ldots1}_{n_2}\underbrace{0\ldots0}_{\ldots}1\ldots$ wherein each bit concerns a step of the measurement [40]. The time operator is established in terms of a multiresolution analysis that concerns dependence on binary digits, which is required by the probability measure $M_j = \left\langle \rho_j \middle| \cdot \right\rangle$. The probability has been defined on the measurable space $\mathfrak{M}_j$ that is generated by single events $P^o_{<j,k}$ of the measurement process, which gives rise to the σ-algebra $\mathfrak{M}$ over the unit interval.

The optimal representation requires finding out wavelets $\psi^o_{j,k}$ such that $P^o_{j,k} = \psi^o_{j,k} \otimes \psi^{o*}_{j,k}$ are generators of $\mathfrak{M}$. The time operator eigenvalues correspond to scales in the wavelet domain hidden Markov model. The statistical model of the measurement claims that detail coefficients are decorrelated in the optimal base. The model regards coefficients to be random variables, which is achieved by an extension that has embedded the process into dyadic line $\mathbb{D}$. In that respect, one should agree to the statement by Koziryev that "wavelet theory is a p-adic spectral analysis" [64].

The information contained in detail coefficient is independent of wavelets, since $H(C\mathbf{D}) = H(\mathbf{D}) + \log|\det C| = H(\mathbf{D})$ for a unitary operator C representing the base substitution. It is decomposed due to the canonical relation $H(\mathbf{D}) = H(\mathbf{S}) + H(\mathbf{D}|\mathbf{S})$ wherein the first term represents the structural information and the second one is an irreducible randomness that persists even if the correlation between coefficients has been recognized [43]. The structural information depends on the base and the optimal base is defined by its maximization which should correspond to the decorrelation requirement [44]. It relates as well to the most significant increase of entropy in the temporal domain, which the measurement is characterized by [62].

Von Neumann made a reference to Bohr, who was the first to have pointed out a link between quantum theory and the principle of psychophysical parallelism [65]. Bohr has adopted Fechner's psychophysics which is termed the *identity view*, since the observer is not to be considered a conglomerate of two substances but one single entity [66]. The most significant sources for psychophysical parallelism are the foreword and the introduction from the *Elements of Psychophysics* [67]. The *outer psychophysics*, which is a link between sensation and stimulation, is realized through the neuroaesthetical computation that relates sensation and neural activity, which is termed by Fechner to be the *inner psychophysics* [41].

A repercussion of von Neumann's solution to the measurement problem is that irreversibility takes place in the presence of observer's mind, which seems to play an active role in the process. The only manner to make such an unpleasant situation compatible with the principle of psychopchysical parallelism requires switching into the inner psychophysics due to a change in representation $\Lambda = \lambda(T)$ which is the operator function of time. The canonical relation separates the inner psychophysical information contained in hidden variables from irreducible randomness having noise properties. An innate component of the wavelet domain hidden Markov model is a denoising procedure that has proven to be advantageous over other methods [42]. It is performed in a superior manner by usage of the optimal base, and regarding that the inner psychophysics corresponds to such a denoising procedure [41]. The optimal measurement is therefore related to recognition of a structure, which should maximize the information it contains [43]. The process is represented by the time operator, constituting the multiresolution analysis [40].

4.2. General measurement

The von Neumann measurement, in terms of mutually orthogonal projectors such that $\sum_{j,k} P_{j,k} = I$ on $L^2_\mu \ominus \mathbb{0}$, concerns a superprojection $\mathcal{M}\rho = \sum_{j,k} P_{j,k}\rho P_{j,k}$ to the subspace of operators commuting with $P_{j,k}$. A generalization corresponds to $\mathcal{M}\rho = \sum_{j,k} Q_{j,k}\rho Q_{j,k}{}^{\dagger}$ which has satisfied $\sum_{j,k} Q_{j,k}{}^{\dagger} Q_{j,k} = I$ in order to preserve the unit trace of a density. The operators might be constructed in the manner of $Q_{j,k} = \left\langle \mathfrak{F}\psi_{j,k} \middle| \cdot \right\rangle \widetilde{\psi_{j,k}}$, wherein $\psi_{j,k}$ is a wavelet frame and $\widetilde{\psi_{j,k}}$ its dual which has been normalized [44]. An operator $\mathfrak{F}$ should map it to the Parseval frame $\mathfrak{F}\psi_{j,k}$.

Frames are geometrically interpreted to be sequences that dilate to bases of an extended space. It means that there is a base which has projected in the initial space to be a frame, whereby a Parseval frame is the projection of an orthonormal base [51]. In that manner, the von Neumann measurement restricts to a general one by neglecting an environment that remains out of the scope. The general measurement should therefore relate to an open system that has been partially described by the stochastic process [68].

The Naimark dilatation theorem implies a method analogous to heterodyne detection in communication engineering: the ensemble to be observed combines with another one which is termed *ancilla* [69]. Thereafter, a von Neumann measurement is performed on the space which has been extended by an environment. The amount of information obtained in that manner might be larger than if the observer is restricted to the von Neumann measurement without ancilla [68]. One concludes that the optimal representation does not restrict to orthonormal wavelets only, since it should involve a general measurement which relates to the frame [44].

Antoniou and Gustafson have pointed out that wavelets and stochastic processes "possess the same common structure so much so that we may regard wavelet theory to be embedded into that of stochastic processes". Such a view should therefore be distinguished from that of other research "in which wavelets are used only to *approximate* stochastic processes in applications" [38]. On the other hand, it has been demonstrated that the measurement, which presents a basic stochastics, is embedded in wavelet theory. The optimal representation concerns a maximization of the structural information which is related to the measurement process. Concerning that, it might be stated that any process comes to be a measurement if addressed in the optimal manner. Palle Jorgenson has discerned that "analogies to Euclidean algorithm seem especially compelling" [45].

The representation intends to recognize an underlying dynamics which has frequently been neglected in both stochastics and signal processing. It is represented by the time operator which gives rise to a multiresolution analysis governing the measurement process. The conception is successfully generalized to frames, as well as to the continuous wavelet transform. In that manner, one has established a complete analogy between wavelet theory and continuous-time regular stationary processes [38].

An underlying dynamics is considered extraordinarily significant in respect to geophysics whereat the concept of wavelet arose [70]. Various wavelets are thought of and used in seismic prospecting, to recognize a pulse which has propagated through the earth [71]. Such a pulse is the wavelet and a primordial assumption concerns that seismic wavelets do not interfere traveling through the earth, which seems to be a decorrelation requirement. The deconvolution has been used to separate a structural information concerning dynamics of the seismic process from a random component [72].

5. Conclusion

The link between wavelets and stochastic processes is extremely complex and interdisciplinary one. Various areas such as geophysics, optics, signal and image processing, quantum theory etc. should not be bypassed. Above everything else, there is a primordial relation to the very origin of geometry that concerns a measurement problem. An emergence of the Euclidean algorithm in that regard takes to a root of real analysis, but as well of p-adic analysis and their interdependence. It might be said without any exaggeration that wavelet theory is reminiscent to the *Mathesis Universalis* which should be a universal calculus involving everything in existence.

A relationship to geophysics seems extraordinarily significant, since the concept of wavelet has arisen in seismic prospecting. In such a context, it is best seen the recognition of an underlying dynamics which wavelets have been intended to. The optimal representation is nothing but a binary code originated by the measurement process which is an intensional procedure producing real numbers of the unit interval. According to that, any process is a measurement if regarded in the optimal manner which involves a recognition of the time operator.

In that respect, detail subspaces of the multiresolution analysis are regarded to be age eigenstates wandering in terms of a bilateral shift dependent on orthonormal wavelets. A change in representation which is the operator function of time has transfigured the reversible evolution to an irreversible one which is the measurement characterized by. The optimal base is defined by maximization of the structural information which relates to the most prominent increase of entropy in the temporal domain. The conception is successfully generalizable to frames which correspond to redundant generators of the Hilbert space.

CRediT authorship contribution statement

Miloš Milovanović: Conceptualization, Investigation, Methodology, Writing – original draft. **Bojan M. Tomić:** Conceptualization, Investigation, Methodology, Writing – original draft. **Nicoletta Saulig:** Validation, Visualization, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

The paper is supported by the Ministry of Science, Technological Development and Innovations of the Republic of Serbia within the research program of the University of Belgrade – Institute for Multidisciplinary Research (Contract No. 451-03-47/2023-01/200053) and of the Mathematical Institute of the Serbian Academy of Sciences and Arts for 2023.

References

[1] Meyer Y. Wavelets: algorithms and applications. Philadelphia: Society for Industrial and Applied Mathematics; 1993.

[2] Srivastava HM, Shah FA, Nayied NA. Fibonacci wavelet method for the solution of the non-linear Hunter–Saxton equation. Appl Sci 2022;12(15):7738.

[3] Srivastava HM, Shah FA, Lone WZ. Quadratic-phase wave-packet transform in $L^2(R)$. Symmetry 2022;14(10):2018.

[4] Srivastava HM, Tantary AY, Shah FA. A new discretization scheme for the non-isotropic stockwell transform. Mathematics 2023;11(8):1839.

[5] Srivastava HM, Shah FA, Lone WZ. Fractional nonuniform multiresolution analysis in $L^2(R)$. Math Methods Appl Sci 2021;44(11):9351–72.

[6] Debnath L, Shah FA. Wavelet transforms and their applications. Boston: Birkhäuser; 2015.

[7] Shah FA, Tantary AY. Wavelet transforms: kith and kin. Boca Raton: CRC Press, Chapman & Hall; 2022.

[8] Srivastava HM, Shah FA, Teali AA. On quantum representation of the linear canonical wavelet transform. Universe 2022;8(9):477.

[9] Hedayati M, Ezzati RR, Noeiaghdam S. New procedures of a fractional order model of novel coronavirus (COVID-19) outbreak via wavelets method. Axioms 2021;10(2):122.

[10] Gaur P, Malaviya V, Gupta A, Bhatia G, Pachori RB, Sharma D. COVID-19 disease identification from chest CT images using empirical wavelet transformation and transfer learning. Biomed Signal Process Control 2022;71:103076.

[11] Kojić M, Schlüter M, Mitić P, Hanić A. Economy-environment nexus in developed European countries: Evidence from multifractal and wavelet analysis. Chaos Solitons Fractals 2022;160:112189.

[12] Lahmiri S, Bekiros S, Bezzina F. Evidence of the fractal market hypothesis in European industry sectors with the use of bootstrapped wavelet leaders singularity spectrum analysis. Chaos Solitons Fractals 2022;165:112813.

[13] Lahmiri S, Bekiros S. Big data analytics using multi-fractal wavelet leaders in high-frequency bitcoin markets. Chaos Solitons Fractals 2020;131:109472.

[14] Agrawal K, Kumar R, Kumar S, Hadid S, Momani S. Bernoulli wavelet method for non-linear fractional Glucose–Insulin regulatory dynamical system. Chaos Solitons Fractals 2022;164:112632.

[15] Guyo G, Pavlov A, Pitsik E, Frolov N, Badarin A, Grubov V, Pavlova O, Hramov A. Cumulant analysis in wavelet space for studying effects of aging on electrical activity of the brain. Chaos Solitons Fractals 2022;158:112038.

[16] Bhardwaj S, Chandrasekhar E, Seemala GK, Gadre VM. Characterization of ionospheric total electron content data using wavelet-based multifractal formalism. Chaos Solitons Fractals 2020;134:112038.

[17] Anvari R, Siahsar MAN, Gholtashi S, Kahoo AR, Mohammadi M. Seismic random noise attenuation using synchrosqueezed wavelet transform and low-rank signal matrix approximation. IEEE Trans Geosci Remote Sens 2017;55(11):6574–81.

[18] Sinitsyn I, Sinitsyn V, Korepanov E, Konashenkova T. Wavelet modeling of control stochastic systems at complex shock disturbances. Mathematics 2021;9(20).

[19] Lemarie PG, Meyer Y. Ondelettes et bases hilbertiennes. Rev Mat Ibero Am 1986;2(1–2):1–18.

[20] Haar A. Zur Theorie der orthogonalen Funktionensysteme. Erste Mitt Math Ann 1910;69(3):331–71.

[21] Graps A. An introduction to wavelets. IEEE Comput Sci Eng 1995;2(2):50–61.

[22] Rognes J. Yves Meyer: restoring the role of mathematics in signal and image processing. 2017, Lecture given on the University of Oslo, Norway, Available online: https://www.mn.uio.no/math/personer/vit/rognes/papers/meyerkoll.pdf.

[23] Heil C, Walnut DF. Fundamental papers in wavelet theory. Princeton and Oxford: Princeton University Press; 2006.

[24] Grossmann A, Morlet J. Decomposition of Hardy functions into square integrable wavelets of constant shape. SIAM J Math 1984;15(4):723–36.

[25] Guan L, Du Y, Li L. Wavelets in petroleum industry: past, present and future. In: SPE annual technical conference and exhibition. Houston; 2004.

[26] Morlet J. Sampling theory and wave propagation. In: 51st annual international meeting of the society of exploration geophysicists. 1981.

[27] Goupillaud P, Grossmann A, Morlet J. Cycle-octave and related transforms in seismic signal analysis. Geoexploration 1984;23:85–102.

[28] Daubechies I. Orthonormal bases of compactly supported wavelets. Comm Pure Appl Math 1988;41(7):909–96.

[29] Addison P. Illustrated wavelet transform handbook introductory theory and applications in science, engineering, medicine and finance. 2nd ed. Bristol and Philadelphia: Institute of Physics Publishing; 2002.

[30] Jaffard S, Meyer Y, Ryan RD. Wavelets – tools for science & technology. Philadelphia: Society for Industrial and Applied Mathematics; 2001.

[31] Hubbard BB. World according to wavelets: the story of a mathematical technique in the making. 2nd ed. CRC Press; 2010.

[32] Meyer Y. Ondelettes et fonctions spline. Séminaire équations aux dérivées partielles (polytechnique). Paris: Ecole Polytechnique; 1986-1987.

[33] Mallat SG. Multiresolution approximations and wavelet orthonormal bases of $L^2(R)$. Trans Amer Math Soc 1989;315(1):69–87.

[34] Gustafson K. Wavelets and expectations: a different path to wavelets. In: Chuong N, Egorov Y, Khrennikov A, Meyer Y, Mumford D, editors. Harmonic, wavelet and p-adic analysis. Hackensack, New York: World Scientific Publishing Co., Inc.; 2007, p. 5–22.

[35] Prigogine I. From being to becoming: time and complexity in the physical science. San Francisco: W.H. Freeman and Company; 1980.

[36] Gustafson K, Misra B. Canonical commutation relations of quantum mechanics and stochastic regularity. Lett Math Phys 1976;1(4):275–80.

[37] Klauder J, Skagerstam B. Coherent states – applications in physics and mathematical physics. Singapore: World Scientific; 1985.

[38] Antoniou I, Gustafson K. Wavelets and stochastic processes. Math Comput Simulation 1999;49(1–2):81–104.

[39] Misra B, Prigogine I, Courbage M. From deterministic dynamics to probabilistic descriptions. Phys A 1979;98(1–2):1–26.

[40] Milovanović M, Vukmirović S, Saulig N. Stochastic analysis of the time continuum. Mathematics 2021;9(12):1452.

[41] Milovanović M, Medić-Simić G. Aesthetical criterion in art and science. Neural Comput Appl 2021;33(6):2137–56.

[42] Crouse MS, Nowak DR, Baraniuk RG. Wavelet-based statistical signal processing using hidden Markov models. IEEE Trans Signal Process 1998;46(4):886–902.

[43] Milovanović M, Rajković M. Quantifying selforganization with optimal wavelets. Europhys Lett 2013;102(4):40004.

[44] Milovanović M. Measurement problem in statistical signal processing. In: 10th international conference on new frontier in physics (ICNFP 2021), Crete. 2021, submitted for publication.

[45] Jorgensen PET. Analysis and probability: wavelets, signals, fractals. Springer; 2006.

[46] Meyer Y. Wavelets and functions with bounded variation from image processing to pure mathematics. Rend Lincei Mat Appl 2000;9:77–105, Special Issue: Mathematics Towards the Third Millennium (Papers from the International Conference Held in Rome).
[47] Milovanović M, Saulig N. An intensional probability theory: investigating the link between classical and quantum probabilities. Mathematics 2022;10(22):4294.
[48] Daubechies I. Ten lectures on wavelets. Philadelphia: Society for Industrial and Applied Mathematics; 1992.
[49] Mallat S. A wavelet tour of signal processing: the sparse way. 3rd ed. Academic Press; 2009.
[50] Kahane JP, Lemarie-Rieusset PG. Fourier series and wavelets. Gordon and Breach Science Publishers; 1996.
[51] Han D, Larson DR. Frames, bases and group representations. Providence: American Mathematical Society; 2010.
[52] Chui CH. An introduction to wavelets. Academic Press; 1992.
[53] Daubechies I. The wavelet transform, time-frequency localization and signal analysis. IEEE Trans Inform Theory 1990;36(5):961–1005.
[54] Hernández E, Weiss GA. A first course on wavelets. CRC Press; 1996.
[55] Bruce AL, Zeidan VM, Berstein DS. What is the Koopman operator? A simplified treatment for discrete time systems. In: Proceedings of 2019 American control conference. ACC, Philadelphia: World Scientific Publishing Co., Inc.; 2019, p. 1912–7.
[56] Gnedenko BV, Kolmogorov AN. Limit distributions for sum of independent random variables. Cambridge: Addison-Wessley Publication Company; 1954.
[57] Antoniou I, Misra B, Suchanecki Z. Time operator: innovation ad complexity. New York: John Wiley and Sons; 2003.
[58] Antoniou I, Suchanecki Z. Non-uniform time operator, chaos and wavelets on the interval. Chaos Solitons Fractals 2000;11:423–35.
[59] Antoniou IE, Gustafson KE. The time operator of wavelets. Chaos Solitons Fractals 2000;11(3):443–52.
[60] Skopina M. p-adic wavelets. Poincare J Anal Appl 2015;2:53–63.
[61] Milovanović M. Dyadic floating point. Mat Vesnik 2021;73(3):149–55.
[62] von Neumann J. Mathematical foundations of quantum mechanics. New York: Princeton University Press; 1955.
[63] Knuth D. The art of computer programming, 3rd ed., vol. 2 – Seminumerical algorithm. Reading, Massachusetts; 1998.
[64] Kozyrev S. Wavelet theory as p-adic spectral analysis. Izv: Math 2002;66(2):367–76.
[65] Bohr N. Wirkunsquantum und Naturbeschreibung. Naturwiss 1929;17:483–6.
[66] Heidelberger M. The mind-body problem in the origin of logical empiricism: Herbert Feigl and psychophysical parallelism. In: Parrini P, Salmon WC, Salmon MH, editors. Logical empiricism: historical and contemporary perspectives. Pittsburgh University Press: Pittsburgh; 2003, p. 233–62.
[67] Fechner GT. Elemente der psychophysik; erster theil. Leipzig: Breitkopf und Härtel; 1860.
[68] Peres A, Wootters WK. Optimal detection of quantum information. Phys Rev Lett 1991;66(9):1119–22.
[69] Helstrom CW. Quantum detection and estimation theory. New York: Academic Press; 1976.
[70] Ricker N. Further developments in the wavelet theory of seismogram structure. Bull Seismol Soc Am 1943;33(3):197–228.
[71] Ricker N. Wavelet contraction, wavelet expansion, and the control of seismic resolution. Geophysics 1953;18(4):769–973.
[72] Webster G. Deconvolution. Tulsa: Society of Exploration Geophysics; 1978.